

Comment on “Weyl versus Conformal Invariance in Quantum Field Theory”

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In a recent paper [6], it is argued that conformal invariance in flat spacetime implies Weyl invariance in a general curved background for unitary theories and possible anomalies in the Weyl variation of scalar operators are identified. We show that these anomalies do not exist.

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Scale and conformal symmetries are essential concepts in quantum field theory. The renormalization group evolution of a Poincaré-invariant quantum field theory is controlled by its dynamical behavior under scale transformations. In the specific case where a field theory is classically scale-invariant, the trace of its classical energy-momentum tensor T_μ^μ is given by

$$T_\mu^\mu = \partial_\mu V^\mu, \quad (1)$$

where V^μ is referred to as the “virial current”.

Although only been proved in two dimensions [1] and perturbatively in four dimensions [2–4], it is believed that a Poincaré-invariant interacting field theory that is scale-invariant but not conformally invariant must be non-unitary. This means that with unitarity, the spacetime symmetry group of a Poincaré-invariant quantum field theory with scale invariance is enhanced to the conformal group, and its energy-momentum tensor can be “improved” to be traceless [5].

In a recent paper [6], it is argued that for unitary theories, conformal invariance in flat spacetime implies local Weyl invariance in a general curved background spacetime. Because of diffeomorphism invariance, a scale transformation of the coordinates and that of the fields in flat spacetime are equivalent to the global Weyl transformations on the metric and fields in a curved spacetime, and hence a quantum field theory with scale invariance in the flat spacetime is globally Weyl invariant when coupled to a general curved background. Thus, conformal invariance provides a link between global and local Weyl invariance in unitary theories.

Also, the Weyl transformation of local scalar operators that correspond to primary operators in the flat limit are identified in [6] and the authors find that there are possible “anomalous terms” in the transformation formulas that prevent some of these operators from transforming covariantly. They argued that these anomalous terms cannot be eliminated based on the constraints originated from the Abelian nature of the Weyl transformations.

Here, we show that the “anomalous terms” found in [6] do not exist. This implies that each primary scalar operator in a unitary conformal field theory corresponds to a Weyl covariant operator in the Weyl invariant theory obtained by coupling the original theory to a general background metric.

It is known that the consequences of symmetries of field theories can be expressed in terms of Ward identities relating Green’s functions. For a conformal field theory, the Ward

identity for primary operators $O(x)$ under an infinitesimal conformal transformation takes the form

$$\hat{\sigma}(x)\langle T_\mu^\mu(x)O(x_1)\cdots O(x_n)\rangle = \sum_i \delta^{(d)}(x-x_i)\langle O(x_1)\cdots (-\Delta\hat{\sigma}(x_i)O(x_i))\cdots O(x_n)\rangle, \quad (2)$$

where $\hat{\sigma}(x) = \frac{1}{d}\partial_\mu\epsilon^\mu(x)$ is the restricted local Weyl rescaling factor with the infinitesimal coordinate change $\epsilon^\mu(x)$ given by

$$\epsilon^\mu(x) = a^\mu + \omega^\mu_\nu x^\nu + cx^\mu + 2(b\cdot x)x^\mu - x^2b^\mu \quad (3)$$

for translation, Lorentz transformations, scale and special conformal transformations, respectively, in d -dimensional flat spacetime. Δ is the Weyl dimension of the operator $O(x)$. When the theory is coupled to a general curved metric $g^{\mu\nu}$, it can be shown that with unitary constraints on the dimensions of operators, $T_\mu^\mu = 0$ as an operator relation [6]. Thus the response of the n -point correlator for $O(x)$ to an infinitesimal Weyl transformation $\delta g_{\mu\nu}(x) = 2\sigma(x)g_{\mu\nu}(x)$ contains only contact terms:

$$\sigma(x)\langle T_\mu^\mu(x)O(x_1)\cdots O(x_n)\rangle = \sum_i \delta^{(d)}(x-x_i)\langle O(x_1)\cdots \delta_\sigma O(x_i)\cdots O(x_n)\rangle, \quad (4)$$

where $\delta_\sigma O(x_i)$ is the variation of the operator $O(x_i)$ under the infinitesimal Weyl transformation.

As described in [6], one must have $\delta_\sigma O \rightarrow -\Delta\hat{\sigma}O$ in the flat limit and $\sigma \rightarrow \hat{\sigma}$. In the special case where O does not contain the metric tensor $g_{\mu\nu}$, $\delta_\sigma O$ must transform covariantly, that is, $\delta_\sigma O = -\Delta\sigma O$. The reason that the Weyl variation of the scalar operator O does not contain terms involving the derivatives of σ is simply because that without the metric tensor, no scalar operator can be formed out of derivatives of σ .

Now, let us consider the general case where O consists of matter fields, the metric tensor and their derivatives. As mentioned above, scale transformations in flat spacetime are equivalent to global Weyl transformations in the curved background. Thus, when $\sigma = c$ with c being a constant, we shall have

$$\delta_{\sigma=c}O = -\Delta cO, \quad (5)$$

from which it follows that under a general Weyl transformation, the operator O transforms either covariantly or as

$$\delta_\sigma O = -\Delta\sigma O + O(\partial\sigma). \quad (6)$$

Note that the first term in the variation (6) is the only permitted term proportional to σ . Terms that violate Weyl covariance are at least of order $\partial\sigma$. Terms such as $\sigma R^2 U$ or $\sigma W^{\mu\nu\alpha\beta} W_{\mu\nu\alpha\beta} U$ (where the shorthand notation R stands for the curvature tensor, the Ricci tensor and the Ricci scalar, $W_{\mu\nu\alpha\beta}$ is the Weyl tensor, and U is a scalar operator with Weyl dimension $\Delta - 4$), referred to as the “anomalous terms” in [6], are not allowed unless the operator O is itself proportional to $R^2 U$ or $W^{\mu\nu\alpha\beta} W_{\mu\nu\alpha\beta} U$.

Then, requiring the vanishing of the commutator $[\delta_{\sigma_1}, \delta_{\sigma_2}]O = 0$, the most general Weyl variation allowed by symmetries and unitarity constraints on the dimensions of operators can be identified. The calculations are straightforward but not very illuminating. The results for relevant and marginal scalar operators in $d \leq 6$ are presented as follows.

For $\Delta \geq \frac{d+2}{2}$ and $\Delta \neq 2n$, $n = 1, 2, 3$,

$$\delta_\sigma O = -\Delta\sigma O + A\Box\sigma, \quad (7)$$

where A is a Weyl covariant scalar with Weyl dimension $\Delta_A = \Delta - 2$. As shown in [6], the new operator O' defined as

$$O' = O + \frac{1}{2(d-1)} RA \quad (8)$$

transforms covariantly as $\delta_\sigma O' = -\Delta\sigma O'$.

Operators with $\Delta = 2n$ are special. For $\Delta = 2$, the variation reads

$$\delta_\sigma O_2 = -2\sigma O_2 + c_1 \Box\sigma. \quad (9)$$

For $\Delta = 4$, we have

$$\delta_\sigma O_4 = -4\sigma O_4 + B\Box\sigma + c_2 R\Box\sigma \quad \text{in } d = 4, 5, \quad (10)$$

with the Weyl dimension 2 operator B transforming according to $\delta_\sigma B = -2\sigma B + c'_1 \Box\sigma$, whereas

$$\delta_\sigma O_4 = -4\sigma O_4 + B\Box\sigma + c_2 R\Box\sigma + c_3 \Box^2\sigma \quad \text{in } d = 6. \quad (11)$$

Note that the term involving $\Box^2\sigma$ is allowed only in $d = 6$. This is due to the fact that under the Weyl variation,

$$\delta_{\sigma_2} \Box^2\sigma_1 = -4\sigma_2 \Box^2\sigma_1 + (d-6)g^{\mu\nu}\nabla_\nu(\Box\sigma_1)\nabla_\mu\sigma_2 - 2\Box\sigma_1\Box\sigma_2 + (d-2)\Box(g^{\mu\nu}\nabla_\mu\sigma_1\nabla_\nu\sigma_2). \quad (12)$$

Thus, if the variation $\delta_\sigma O_4$ contains the term involving $\Box^2\sigma$, the commutativity of Weyl transformations cannot be satisfied unless $d = 6$.

Finally, for $\Delta = d = 6$, we have

$$\delta_\sigma O_6 = -6\sigma O_6 + A'\square\sigma + B'\square^2\sigma + B''R\square\sigma + c_4 R^2\square\sigma, \quad (13)$$

where the Weyl variations of the operators A' , B' and B'' are given, respectively, by

$$\delta_\sigma A' = -4\sigma A' + B'''\square\sigma + c_5 R\square\sigma \quad (14)$$

with $\delta_\sigma B''' = -2\sigma B''' + c_1'''\square\sigma$,

$$\delta_\sigma B' = -2\sigma B', \quad (15)$$

and

$$\delta_\sigma B'' = -2\sigma B'' + c_1''\square\sigma. \quad (16)$$

Introducing the operators

$$O'_2 \equiv O_2 + \frac{c_1}{2(d-1)}R, \quad (17)$$

$$O'_4 \equiv O_4 + \frac{1}{2(d-1)}RB + \frac{1}{4(d-1)}(c_2 + \frac{c'_1}{2(d-1)})R^2 \quad \text{in } d = 4, 5, \quad (18)$$

$$O''_4 \equiv O_4 + \frac{1}{10}RB + \frac{1}{20}(c_2 + \frac{c'_1}{10} - \frac{c_3}{5})R^2 + \frac{c_3}{10}\square R \quad \text{in } d = 6, \quad (19)$$

and

$$O'_6 \equiv O_6 + \frac{1}{10}A'R + \frac{1}{20}\left(-\frac{1}{5}B' + B'' + \frac{1}{10}B'''\right)R^2 + \frac{1}{10}B'\square R + \frac{1}{30}\left(c_4 + \frac{c_5}{10} + \frac{c''_1}{20} + \frac{c'''_1}{200}\right)R^3, \quad (20)$$

it is straightforward to show that O'_2 , O'_4 , O''_4 and O'_6 all transform covariantly under an infinitesimal Weyl transformation.

With these results, we conclude that when a conformal field theory in $d \leq 6$ is coupled to a general curved background metric $g_{\mu\nu}$, every primary scalar operator $O(x)$ that is either relevant or marginal corresponds to a Weyl covariant operator $O'(x)$ such that $O'(x) \rightarrow O(x)$ in the flat limit, and operators $O'(x)$ obey the infinitesimal form of the Ward identity for Weyl invariance:

$$\sigma(x)\langle T_\mu^\mu(x)O'(x_1)\cdots O'(x_n)\rangle = \sum_i \delta^{(d)}(x-x_i)\langle O'(x_1)\cdots (-\Delta\sigma(x_i)O'(x_i))\cdots O'(x_n)\rangle. \quad (21)$$

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